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Information entropy of a time-dependent three-level trapped ion interacting with a laser field

Mahmoud Abdel-Aty

Mathematics Department, Faculty of Science, South Valley University, 82524 Sohag, Egypt

E-mail: abdelatyquant@yahoo.co.uk

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Abstract

Trapped and laser-cooled ions are increasingly used for a variety of modern high-precision experiments, frequency standard applications and quantum information processing. Therefore, in this communication we present a comprehensive analysis of the pattern of information entropy arising in the time evolution of an ion interacting with a laser field. A general analytic approach is proposed for a three-level trapped-ion system in the presence of the time-dependent couplings. By working out an exact analytic solution, we conclusively analyse the general properties of the von Neumann entropy and quantum information entropy. It is shown that the information entropy is affected strongly by the time-dependent coupling and exhibits long time periodic oscillations. This feature attributed to the fact that in the time-dependent region Rabi oscillation is time dependent. Using parameters corresponding to a specific three-level ionic system, a single beryllium ion in a RF-(Paul) trap, we obtain illustrative examples of some novel aspects of this system in the dynamical evolution. Our results establish an explicit relation between the exact information entropy and the entanglement between the multi-level ion and the laser field. We show that different nonclassical effects arise in the dynamics of the ionic population inversion, depending on the initial states of the vibrational motion/field and on the values of Lamb–Dicke parameter η.

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(Some figures in this article are in colour only in the electronic version)

Laser-cooled ions confined in an electromagnetic trap are good candidates for various quantum-state engineering processes. Although trapped ions have found many applications in physics [1–3], they caused a turning point in the evolution of quantum computing. Efforts to realize experimentally the elements of quantum computation using trapped atomic ions have been stimulated largely by a proposal of Cirac and Zoller [4]. This proposal also launched an avalanche of other physical realizations of quantum computing using different physical systems, from high finesse cavities to widely manufactured semiconductors [5, 6]. The various theoretical schemes for generating the nonclassical states of motion of the trapped ions and achieving quantum computing with trapped ions are based on the two-level model [7], in which the ion is simplified to be of two levels, the trap's potential is quantized as a harmonic oscillator and the radiating lasers are supposed to be classical forms of standing or travelling waves. The general consideration is taken for the case of Lamb–Dicke regime under the weak excitation regime, which corresponds to the actual case in the present ion-trap experiments [8]. For this case, some techniques developed in the framework of cavity QED based on the two-level model can be immediately transcribed to the ion-trap system by taking advantage of the analogy between the cavity QED and the ion-trap problem.

It appears that a scalable ion-trap system must incorporate arrays of interconnected traps, each holding a small number of ions. Quantum-state transfer and entanglement distribution among distant nodes in a quantum network have been reported in [9]. From this study, one can say that the information carriers between traps might be photons or ions that are moved between traps in the array. In the latter case, a 'head' ion held in a movable trap could carry the information by moving from site to site as in the proposal of Cirac and Zoller [11]. Similarly, as has been proposed at NIST, one could shuttle ions around in an array of interconnected traps [12, 13]. In this last scheme, the idea is to move ions between nodes in the array by applying time-dependent potentials to 'control' electrode segments.

On the other hand, one of the major challenges in the field of quantum information theory is to get a deep understanding of how local operations assisted by classical communication performed on a multi-level quantum system can affect the entanglement between the spatially separated systems. Despite a lot of progress in the last few years, it is still not fully understood. For instance, even for the simple question of whether a given state is entangled or not, no general answer is known [14]. An interesting question raised in [15] is whether there is any relationship between the uncertainty principle and entanglement or not. Recently, a general definition of entropy squeezing for a two-level system has been presented [16] and showed that the information entropy is a measure of the quantum uncertainty of atomic operators. Also, the number-phase entropic uncertainty relation for the multiphoton coherent state and nonlinear coherent state is studied and compared with an ordinary coherent state [17].

In all the previous studies, the quantum information entropy (entropy squeezing) was presented in the two-level system only. When three-level systems were considered, the atom–field coupling is taken to be time independent and the quantum field entropy is investigated [18]. Therefore, it will be of great interest to fill the gap, and to see whether a similar result holds when the multi-level systems are considered. In this paper, we establish the theory of the quantum information entropy in the time-dependent case for a three-level trapped ion. Our purpose is to address some quite general questions about quantum information physics and ion traps, with the aim of identifying useful directions for theoretical and experimental research in the near future and the longer term. Based on the exact conditional quantum dynamics for the laser–ion interaction, an analytic approach is proposed for a three-level trapped ion in the presence of any form of the time-dependent ion–field couplings. A subsequent part of this paper is concerned with finding the general forms of the information entropies for any three-level system.

The present paper is structured as follows: as a necessary introduction, we start by introducing our Hamiltonian model and give an exact analytic solution for the Schrödinger equation in the frame of the dressed state formalism. In section 3 we employ the analytical results obtained in section 2 to demonstrate the main claim, by showing that there exists a unique form of the information entropy in the three-level system, and classify the behaviour in several parameter regimes assuming that the field is in a coherent state. Finally, a summary of the main points of this work ends the paper and a few avenues for further investigations are indicated in section 4.

1. Model

1.1. The Hamiltonian

Accurate potentials are, of course, required for a quantitatively correct prediction of the behaviour and properties of real quantum systems. However, even qualitative conclusions drawn from simulations employing inaccurate or invalidated potentials can be problematic. The most appropriate form of the potential depends largely upon the properties of interest to the simulators. To set the stage, we first begin with a discussion of the basic equations of the three-level trapped-ion model. Therefore, the physical system on which we focus is a three-level harmonically trapped ion with its centre-of-mass motion quantized. We denote by $\hat{a}$ and $\hat{a}^\dagger$ the annihilation and creation operators respectively, and υ is the vibrational frequency related to the centre-of-mass harmonic motion along the direction $\hat{x}$. The Hamiltonian for the trapped ion interacting with laser light may be written as [19, 20]

$$\begin{aligned}\hat{H} &= \hat{H}_0 + \hat{H}_{\text{int}}(t),\\ \hat{H}_0 &= \hbar\upsilon\hat{a}^\dagger\hat{a} + \hbar\omega_a\hat{S}_{aa} + \hbar\omega_b\hat{S}_{bb} + \hbar\omega_c\hat{S}_{cc},\\ \hat{H}_{\text{int}}(t) &= \hbar\lambda_1 E_1^-(\hat{x},t)\hat{S}_{ac} + \hbar\lambda_2 E_2^-(\hat{x},t)\hat{S}_{bc} + \text{h.c.},\end{aligned} \tag{1}$$

where the transition in the three-level ion is characterized by the dipole or quadruple coupling matrix element λ_j. We denote by $\hat{S}_{lm}$ the atomic flip operator for the $|m\rangle \to |l\rangle$ transition between the two electronic states, where $\hat{S}_{lm} = |l\rangle\langle m|(l, m = a, b, c)$. So far we have disregarded relaxations since we are interested in the dynamics for short times. In equation (1), the classical electric field of the driving laser is given by

$$E_j^-(\hat{x},t) = E_0\exp(\text{i}\omega_j t)\cos\left(\frac{\omega_j}{c}\hat{x} + \phi_j\right), \tag{2}$$

where $E_0\exp(\text{i}\omega_j t)$ is the negative frequency part of the classical electric field of the driving laser of amplitude E_0 and frequency ω_l. The operator-valued function $\cos\left(\frac{\omega_l}{c}\hat{x} + \phi_j\right)$ describes the mode structure of the laser field for a standing wave, where ϕ_j defines the position of the trap potential with respect to the wave. The operator $\hat{x}$ is the position operator associated with the centre-of-mass motion. Therefore, we can express the centre-of-mass position in terms of the creation and annihilation operators of the one-dimensional trap, namely

$$\hat{x} = \frac{\eta c}{\omega_l}(\hat{a}^\dagger + \hat{a}), \tag{3}$$

where η is the Lamb–Dicke parameter. For the sake of simplicity (but without loss of generality), we have assumed to deal with the case in which $\phi_1 = \phi_2 = \phi$ and the level $|c\rangle$ is assumed to be dipole-coupled to both the levels $|a\rangle$ and $|b\rangle$ via a far detuned laser field. When the ion is in the resolved sideband limit and the laser is irradiated resonantly to the kth vibrational sideband, we may write the Hamiltonian $\hat{H}_{\text{int}}$, in the interaction picture as follows:

$$\hat{H}_{\text{int}} = \hbar\gamma_1(t)E_0\hat{f}_k^{(1)}(\hat{a}^\dagger\hat{a})\hat{S}_{ac}\hat{a}^{\dagger k} + \hbar\gamma_2(t)E_0\hat{f}_k^{(2)}(\hat{a}^\dagger\hat{a})\hat{S}_{bc}\hat{a}^{\dagger k} + \text{h.c.}, \tag{4}$$

with new time-dependent coupling parameter $\gamma_i(t)$. The above Hamiltonian describes the nonlinear, k-quantum coupling of the vibrational mode and the electronic transition, assisted by the laser field. The other contributions are rapidly oscillating with frequency ν and have been disregarded. Note that in the Lamb–Dicke regime only processes with $k = 0, 1$ are considered, while in the general case, the nonlinear coupling function is derived by expanding the operator-valued mode function. For a standing wave $\hat{f}_k^{(j)}(\hat{a}^\dagger\hat{a})$ is given by

$$\hat{f}_k^{(j)}(\hat{a}^\dagger\hat{a}) = \frac{1}{2}\exp\left(\mathrm{i}\phi_j - \frac{\eta^2}{2}\right)\sum_{n=0}^{\infty}\frac{(\mathrm{i}\eta)^{2n+k}}{n!(n+k)!}\hat{a}^{\dagger n}\hat{a}^n + \text{h.c.} \tag{5}$$

In the time-independent case, the analysis of such a Hamiltonian model can be carried out, providing elimination of the nonresonantly coupled atomic level $|c\rangle$ adiabatically in the same manner as the standard three-level systems. Indeed, due to the large detuning, the transitions for instance from the level $|a\rangle$ to the level $|c\rangle$ are very fast and are immediately followed by decays on the atomic level $|b\rangle$. Therefore, considering only coarse grained observables, meaning that the system is observed at a rough enough time scale, effectively eliminates the far detuned level; namely, at such a time scale, the only observables and hence meaningful dynamical behaviours involve levels $|a\rangle$ and $|b\rangle$ as a result of time averaging second-order processes having $|c\rangle$ as an intermediate virtual level. This procedure hence suppresses the fine dynamics; that is it sacrifices any information concerning the fast dynamics the third level is involved in.

1.2. An analytic solution

There are different ways to solve the system of equations which are obtained from solving Schrödinger equation. One may assume that the three eigenstates of H_0 are known, along with their corresponding eigenenergies [21–23]. The total wavefunction may be expanded in terms of the known eigenstates, namely

$$|\Psi(t)\rangle = \sum_{n=0}^{\infty}(A_1(n,t)|\xi_1\rangle + A_2(n,t)|\xi_2\rangle + A_3(n,t)|\xi_3\rangle), \tag{6}$$

where $(|\xi_1\rangle, |\xi_2\rangle, |\xi_3\rangle) = (|n, c\rangle, |n{+}k, a\rangle, |n{+}k, b\rangle)$. With atomic units, using the Schrödinger equation, we obtain the coupled equations for our three-level system, namely

$$\mathrm{i}\frac{\partial A_j(n,t)}{\partial t} = \sum_{k=1}^{3} H_{jk}A_k(n,t), \tag{7}$$

where $H_{jk} = \langle\xi_j|\hat{H}_{\text{int}}|\xi_k\rangle$. These equations are exact for any three-level system. In order to solve equation (7), we assume that

$$G(n,t) = A_1(n,t) + xA_2(n,t) + yA_3(n,t), \tag{8}$$

which means that

$$\mathrm{i}\frac{\mathrm{d}G(n,t)}{\mathrm{d}t} = (H_{11} + xH_{21} + yH_{31})\left(\frac{H_{12} + xH_{22} + yH_{32}}{H_{11} + xH_{21} + yH_{31}}A_2(n,t) + \frac{H_{13} + xH_{23} + yH_{33}}{H_{11} + xH_{21} + yH_{31}}A_3(n,t) + A_1(n,t)\right). \tag{9}$$

Let us emphasize that in addition to the general form of equation (9), the present method is suitable for any initial states. For a particular case, if we consider the model presented in this paper, then the non-vanishing terms in equation (9) are H_{21}, H_{31} and their complex conjugates, while the rest of $H_{ij} = 0$. In this case H_{21} and H_{31} are given by

$$H_{j1} = \hbar\gamma_{(j-1)}(t)E_0 f_k^{(j-1)}(n)\sqrt{\frac{(n+k)!}{n!}} = (H_{1j})^*, \qquad j = 2, 3. \tag{10}$$

It is instructive to examine the formation of a general solution of the three-level systems. Therefore, we use equation (9) with $\gamma_j(t) = \beta_j \gamma(t)$ and seek $G(n, t)$ such that

$$\mathrm{i}\frac{\mathrm{d}G(n,t)}{\mathrm{d}t} = z\gamma(t)G(n,t). \tag{11}$$

This holds if

$$x = \frac{H_{12} + xH_{22} + yH_{32}}{H_{11} + xH_{21} + yH_{31}}, \qquad y = \frac{H_{13} + xH_{23} + yH_{33}}{H_{11} + xH_{21} + yH_{31}}, \qquad z = H_{11} + xH_{21} + yH_{31}. \tag{12}$$

We consider an initial state of the system in which the vibrational phonon subsystem is in a coherent state and the ion is in an upper state. After some algebra this leads to a cubic equation which has three eigenvalues $x_i(y_i)$ which determine the z_i. There are also three corresponding eigenfunctions

$$G_j(n,t) = G_j(0)\exp\left(-\mathrm{i}z_j\int_0^t \gamma(t)\,\mathrm{d}t\right).$$

Then, one can obtain

$$G_j(n,t) = M_{j1}A_1(n,t) + M_{j2}A_2(n,t) + M_{j3}A_3(n,t), \tag{13}$$

where

$$M_{ji} = \hat{k}^*\hat{e}_x + \hat{x}^*\hat{e}_y + \hat{y}^*\hat{e}_z, \tag{14}$$

with $\hat{e}_x$, $\hat{e}_y$ and $\hat{e}_z$ being mutually orthogonal unit vectors, given by $\hat{e}_x = (1, 0, 0)$, $\hat{e}_y = (0, 1, 0)$ and $\hat{e}_z = (0, 0, 1)$. Also, $\hat{k} = (1, 1, 1)$, $\hat{x} = (x_1, x_2, x_3)$ and $\hat{y} = (y_1, y_2, y_3)$, where the asterisk means that the row vector becomes column vector.

Now, we express the unperturbed state amplitude $A_1(n, t)$, $A_2(n, t)$ and $A_3(n, t)$ in terms of the dressed state amplitude $G_j(n, t)$,

$$\begin{aligned} A_i(n,t) &= \sum_{j=1}^{3} M_{ij}^{-1} G_j(n,t) \\ &= \sum_{j=1}^{3} M_{ij}^{-1} G_j(0)\exp\left(-\mathrm{i}z_j\int_0^t \gamma(t)\,\mathrm{d}t\right). \end{aligned} \tag{15}$$

Using the above equations, we can obtain

$$A_j(n,t) = \frac{1}{D}\sum_{m=1}^{3} A_{jm}(n,t)\exp\left(-\mathrm{i}z_j\int_0^t \gamma(t)\,\mathrm{d}t\right), \tag{16}$$

where

$$\begin{aligned} &A_{11}(n,t) = x_2y_3 - y_2x_3, \qquad A_{12}(n,t) = x_3y_1 - y_3x_1, \qquad A_{13}(n,t) = x_1y_2 - y_1x_2, \\ &A_{21}(n,t) = y_2 - y_3, \qquad A_{22}(n,t) = y_3 - y_1, \qquad A_{13}(n,t) = y_1 - y_2, \\ &A_{31}(n,t) = x_2 - x_3, \qquad A_{32}(n,t) = x_1 - x_3, \qquad A_{33}(n,t) = x_2 - x_1, \\ &D = x_1y_2 + x_2y_3 + x_3y_1 - x_1y_3 - x_2y_1 - x_3y_2. \end{aligned} \tag{17}$$

We have thus completely determined the exact solution of a time-dependent three-level system. This analytic solution does not depend on the form of the time-dependent ion–field couplings, although it is included by the proper definition of our system. This gives us a good possibility of estimating the nonclassical behaviour of different three-level systems. In a special situation, where we assume that $\gamma(t) = 1$, the nonlinear k-quantum Rabi frequency can be written as

$$\mu_n = 2E_0(\mathrm{i}\eta)^k L_n^k(\eta^2)t\sqrt{\left(\beta_1^2 + \beta_2^2\right)\frac{(n+k)!}{n!}}, \tag{18}$$

where $L_n^k(\eta^2)$ is an associated Laguerre polynomial. The picture in this case is of the three-level system in the presence of the time-dependent ion–field coupling, rather than the usual picture of the three-level system. The important point to note here is that, using the above analytic approach, any three-level Hamiltonian is likewise exactly solvable, with precisely similar eigenvectors and eigenvalues that are obtained directly using equations (4) and (6).

It is interesting to compare our results with those of the previous work. When we neglect the time dependence of the ion–field couplings and consider the single-photon transition, we arrive at an equation similar to equation (21) of reference [24] which has been obtained using the unitary transformation method. Our formulation is based on the conventional Schrödinger equation, but is distinguished from other treatments by the inclusion of time-dependent ion–field couplings, intensity-dependent and multi-photon interaction. These features make the model more general than the previous studies.

2. Quantum information entropy

2.1. Shannon entropy

Let us focus more specifically on the first case of interest for this paper, Shannon information which is defined to solve the problem of the most efficient coding of a set of signals. In the synthesis of probability distribution, Shannon's entropy has played an important role in the study of quantum-mechanical systems and clarifying the fundamental concepts. In an analogous way, the Shannon information entropy corresponding to the photon number distribution [25] is defined by

$$S_H(t) = -\sum_{n=0}^{\infty} P(n,t)\ln P(n,t), \tag{19}$$

where $P(n,t)$ is the photon number distribution $P(n,t) = \langle n|\hat{\rho}_F|n\rangle$, where ρ_F is the reduced density matrix of the vibrational mode. The information quantity that results from a measurement is still defined in terms of Shannon information on the measurement outcomes. This depends upon the particular measurement that is performed.

For the system under consideration when the ion is initially at its upper level $|c\rangle$, $P(n,t)$ is given by

$$P(n,t) = \sum_{n=0}^{\infty}(|A_1(n,t)|^2 + |A_2(n-1,t)|^2 + |A_3(n-1,t)|^2). \tag{20}$$

We now discuss the results obtained numerically and the interesting situation occurring for different forms of the ion–field coupling. For applications in real systems, we consider $^9Be^+$ and the commonly used state as initial condition for the field: the coherent state. A coherent state of motion $|\alpha\rangle$ of the ion corresponds to a minimum uncertainty wave-packet whose centre oscillates classically in the harmonic well and retains its shape. The probability distribution among Fock states is Poissonian,

$$P_n = |\langle n|\alpha\rangle|^2 = \exp(-\overline{n})\frac{\overline{n}^n}{n!}, \tag{21}$$

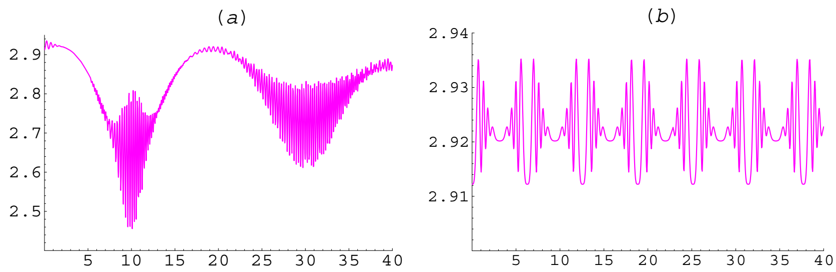

Figure 1. Time dependence of the Shannon entropy for $\eta = 0.202$, $\bar{n} = 20$ and for different forms of the ion–field coupling, where (*a*) $\gamma(t) = \gamma$ and (*b*) $\gamma(t) = \gamma\sin(\varpi t)$.

with $\overline{n} = |\alpha|^2$. In the typical experiments at NIST [26], a single $^9Be^+$ ion is stored in a Paul trap with a secular frequency along x of $\nu/2\pi \simeq 11.2$ MHz, providing a spread of the ground-state wavefunction of $x \simeq 7$ nm, with a Lamb–Dicke parameter of $\eta \simeq 0.202$. The laser beam, with 0.5 W, is approximately detuned $\Delta/2\pi \simeq 12$ GHz, so that $\Omega/2\pi \simeq 475$ kHz. With these data we find $E_0 \simeq 0.01$, so it can be considered as a small parameter. The Shannon information entropy $S_H(t)$ corresponds to the photon number distribution in which the reduced density matrix for the field subsystem is shown in figure 1, when the initial state of the laser field is a coherent state and the trapped ion starts from its upper level, where the ion–field couplings are time independent, i.e. $\gamma(t) = \gamma$ in figure 1(*a*) and time dependent, i.e. $\gamma(t) = \gamma\sin(\varpi t)$ in figure 1(*b*). Note that $S_H(t)$ not only increases but also exhibits a well-marked oscillatory behaviour, indicating that the system recovers coherence in a periodic way, for sufficiently short times.

Our goal is to uncover interesting relations between the periodic oscillations and that emerge when we study different forms of the ion–field couplings from the viewpoint of information entropy. In figure 1(*b*), we consider the time-dependent ion–field couplings. In this case, the general behaviour of the Shannon information entropy is completely periodic. If the argument is valid, one should expect the existence of these periodic oscillations for a large period of the interaction. This is, in fact, the opposite to the case in figure 1(*a*) where one sees, moreover, that it takes a longer time for $S_H(t)$ to reach the plateau, due to the considerable time-independent couplings between the ion and the laser field. More specifically, by taking the ion–field coupling to be time dependent we obtain a perfect periodic oscillation of the $S_H(t)$, which is due to the appearance of the factor $\cos(\varpi t)$ in the Rabi frequency μ_n in the time-dependent case.

2.2. Quantum field entropy

So far we have made a critical assumption in analysing the information gained from measurements, namely that measurements have well-defined outcomes, and that we have a clear understanding of when and how a measurement has been occurred. This is, of course, a deeply controversial aspect of the interpretation of quantum theory. On the other hand, the standard von Neumann definition of the quantum-mechanical entropy [27] is given by

$$S = -\mathrm{Tr}\{\hat{\rho}\ln\hat{\rho}\}, \tag{22}$$

where $\hat{\rho}$ is the density operator for a given quantum system and we have set Boltzmann's constant $K = 1$.

The von Neumann entropy is of central importance in physics; when applied to a thermal ensemble, it is the entropy of thermodynamics. In quantum information theory it plays

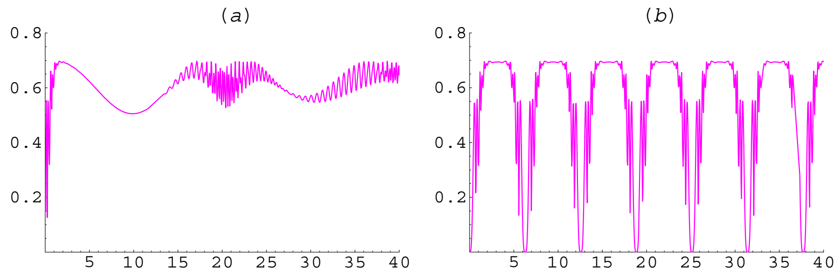

Figure 2. The time evolution of the quantum field entropy $S_f(t)$, the initial average photon number $\bar{n} = 20$, and $\eta = 0.202$ (*a*) time-independent ion–field coupling $\gamma(t) = \gamma$ and (*b*) time-dependent case, $\gamma(t) = \gamma\sin(\varpi t)$.

prominent roles in many contexts, e.g., in studies of the classical capacity of a quantum channel and the compressibility of a quantum source [28]. If $\hat{\rho}$ describes a pure state, then $S = 0$, and if $\hat{\rho}$ describes a mixed state, then $S \neq 0$. In the pure state case, Phonix and Knight succeeded in evaluating the field entropy in a closed form and showed that it did indeed equal the atomic entropy at all times. The entropies of the atom and the field, when treated as a separate system, are defined through the corresponding reduced density operators by

$$S_{A(F)} = -\mathrm{Tr}_{A(F)}\{\hat{\rho}_{A(F)} \ln \hat{\rho}_{A(F)}\}. \tag{23}$$

In order to derive a calculation formalism of the field entropy, we must obtain the eigenvalues of the reduced field density operator. Since the trace is invariant under the similarity transformation, we can go to the basis in which the atomic density matrix ρ_A is diagonal and since the system is closed [29],

$$S_F = -\mu_1 \ln \mu_1 - \mu_2 \ln \mu_2 - \mu_3 \ln \mu_3, \tag{24}$$

where μ_j are the eigenvalues of the atomic reduced density matrix which satisfy the third-order equation

$$\mu^3 - \mu^2 + \delta_1 \mu + \delta_2 = 0, \tag{25}$$

where

$$\begin{aligned}\delta_1 &= \rho_{33}\rho_{22} + \rho_{22}\rho_{11} + \rho_{11}\rho_{33} - |\rho_{12}|^2 - |\rho_{23}|^2 - |\rho_{31}|^2,\\ \delta_2 &= -\rho_{33}\rho_{22}\rho_{11} - 2\,\mathrm{Re}(\rho_{12}\rho_{23}\rho_{31}) + \rho_{11}|\rho_{23}|^2 + \rho_{22}|\rho_{13}|^2 + \rho_{33}|\rho_{12}|^2.\end{aligned} \tag{26}$$

Equation (25) is expected to have three different real roots. They are given by

$$\mu_j = \tfrac{1}{3} + \tfrac{2}{3}\left(\sqrt{1 - 3\delta_1}\right)\cos(\theta_j), \tag{27}$$

where

$$\theta_j = \left(\frac{1}{3}\arccos\left[\frac{-9\delta_1 + 2 - 27\delta_2}{2(1 - 3\delta_1)^{\frac{3}{2}}}\right] + (j-1)\frac{2\pi}{3}\right), \qquad j = 1, 2, 3. \tag{28}$$

The numerical results of the von Neumann entropy given by equations are shown in figure 2. As stated above, we deal in this paper with two kinds of ion–field coupling, and for numerical simulations we consider a single $^9Be^+$ ion is stored in a Paul trap. To study the behaviour of the field entropy as a function of the scaled time γt, we have initially fixed the mean photon numbers $\overline{n} = 20$. We now consider both the centre-of-mass motion and the field initially prepared in coherent states and the ion in its upper state. The Lamb–Dicke parameter is typically less than unity, and the lowest order term in the expansion above is of second order in η (say $\eta = 0.2$). In our treatment we have considered different cases; however, for the sake of comparison we have taken at the beginning the time-independent case, where we can see the usual behaviour of the field entropy in figure 2(*a*). We may mention that there are two types of Rabi oscillations, one at a smaller amplitude than the other. It is remarked that the first maximum of the field entropy at $t > 0$ is achieved at the collapse time, and at one-half of the revival time, the entropy reaches its local minimum. In this case we observe that the ion and the field are disentangled at the beginning, but as time develops the coupling starts to play its role. However, it takes a long period before it reaches the strongest entanglement and then this strong entanglement is sustained for a long period before fluctuations start to appear. This behaviour is drastically changed as soon as the ion–field coupling is taken to be time dependent (see figure 2(*b*)). Similar to the behaviour of Shannon entropy, in this case, we see that, as soon as we consider the time-dependent case, the period of revivals becomes shorter and the time interval of vibration of the entropy is compressed (see figures 1(*b*), 2(*b*)). Here we can observe the regular behaviour of the entropy. This is due to the periodicity of the $f(\varpi t)$ function.

Some remarks on the periodic evolutions are now in order. For subsystems with an available Hamiltonian, the consideration of the time-dependent ion–field coupling shows a perfect periodic oscillation. It is in this sense that we conclude that the entropy, purity gains found in the time evolution of a three-level trapped ion interacting with a laser field, are related to the considered ion–field coupling forms. Also, the initial state of the field plays an important role of the established connection between entropy and entanglement.

It is interesting to note that when we put $k = 1$, $\hat{f}_k^{(j)}(\hat{a}^\dagger\hat{a}) = 1$ and $\gamma(t) = \gamma$ (time-independent case) we get the results of [18]. Another point worth noting in this context is the possibility of manipulating different forms of the nonlinearity involved in the present model.

2.3. *Atomic information entropy*

Although the foregoing schemes using atomic entropy represent advances by simplifying the required procedures to measure the entanglement, all the investigations implemented until now consider only a quantum field entropy equal to the atomic entropy, due to the assumption that the system starts from a pure state. However, atomic information entropy via the atomic operators is an important scenario for testing fundamentals of entanglement behaviour as well as for demonstrating quantum information processing, even if the system starts from a mixed state.

In what follows, we propose an oversimplified scheme to derive the analytical formulae of the information entropy of three-level system in terms of the eigenvalues and eigenvectors of the atomic operators $\hat{S}_x$, $\hat{S}_y$ and $\hat{S}_z$. In the three-level system, the operators $\hat{S}_x$, $\hat{S}_y$ and $\hat{S}_z$ form a spin-1 representation of angular momentum group, and can be written in the following form [30]:

$$\begin{aligned} \hat{S}_x &= (\hat{e}_y^*\hat{e}_x + \hat{e}_x^*\hat{e}_y + \hat{e}_z^*\hat{e}_y + \hat{e}_y^*\hat{e}z)/\sqrt{2}, \\ \hat{S}_y &= (-\mathrm{i}\hat{e}_y^*\hat{e}_x + \mathrm{i}\hat{e}_x^*\hat{e}_y - \mathrm{i}\hat{e}_z^*\hat{e}_y + \mathrm{i}\hat{e}_y^*\hat{e}_z)/\sqrt{2}, \\ \hat{S}_z &= -\hat{e}_x^*\hat{e}_x + \hat{e}_z^*\hat{e}_z. \end{aligned} \tag{29}$$

The mutually orthogonal unit vectors $\hat{e}_x$, $\hat{e}_y$ and $\hat{e}_z$ are given in equation (14). Also, the operators $\hat{S}_i$ can be defined in terms of the operators $\hat{S}_{ij}$.

The probability distribution of the three possible outcomes of measurements of an operator $\hat{S}_\alpha$, in this case, is defined by

$$P_i(\hat{S}_\alpha) = \langle\psi_{\alpha_i}|\hat{S}_\alpha|\psi_{\alpha_i}\rangle, \qquad \alpha = x, y, z \qquad \text{and} \qquad i = 1, 2, 3 \tag{30}$$

where $|\psi_{\alpha_i}\rangle$ are the eigenvectors of the matrix $(\hat{S}_\alpha - \beta I) = 0$, and β are the associated eigenvalues. Thus,

$$\begin{aligned} P_1(\hat{S}_x) &= \tfrac{1}{2}(1-\rho_{22}) - \mathrm{Re}[\rho_{13}], \\ P_2(\hat{S}_x) &= \tfrac{1}{4}(1+\rho_{22}) + \tfrac{1}{2}\,\mathrm{Re}[\rho_{13} - (\rho_{12}+\rho_{23})\sqrt{2}], \\ P_3(\hat{S}_x) &= \tfrac{1}{4}(1+\rho_{22}) + \tfrac{1}{2}\,\mathrm{Re}[\rho_{13} + (\rho_{12}+\rho_{23})\sqrt{2}]. \end{aligned} \tag{31}$$

The probability distribution of the three possible outcomes of measurements of an operator $\hat{S}_y$

$$\begin{aligned} P_1(\hat{S}_y) &= \tfrac{1}{2}(1-\rho_{22}) + \mathrm{Re}[\rho_{13}], \\ P_2(\hat{S}_y) &= \tfrac{1}{4}(1+\rho_{22}) - \tfrac{1}{2}\,\mathrm{Re}[\rho_{13}] - \tfrac{1}{\sqrt{2}}\,\mathrm{Im}(\rho_{12}+\rho_{23}), \\ P_3(\hat{S}_y) &= \tfrac{1}{4}(1+\rho_{22}) - \tfrac{1}{2}\,\mathrm{Re}[\rho_{13}] + \tfrac{1}{\sqrt{2}}\,\mathrm{Im}(\rho_{12}+\rho_{23}). \end{aligned} \tag{32}$$

Finally, the probability distribution of $\hat{S}_z$

$$P_i(\hat{S}_z) = \rho_{ii}.$$

Using the above equations we can write the information entropies $H(\hat{\sigma}_\alpha)$, $\alpha = x, y, z$, in the following form:

$$H(\hat{\sigma}_\alpha) = -\sum_{i=1}^{3} P_i(\hat{S}_\alpha)\ln P_i(\hat{S}_\alpha), \qquad \alpha = x, y, z. \tag{33}$$

To avoid misunderstandings, we want to remind the reader that many entropy based separability criteria are known, which relate the entropy of the total state with the entropy of its reductions. The main difference between this approach and ours is that in our approach the probability distribution of the outcomes of a measurement is taken into account, and not the eigenvalues of the density matrix. Our criteria can therefore directly be applied to measurement data; no state reconstruction is needed. Although we derived a formal expression for the quantum information entropy, it is almost impossible to learn something by inspection from it. Thus, we numerically study the quantity $H(\hat{\sigma}_\alpha)$ as a function of the scaled time. We consider that the initial state of the field is a coherent state which is linear superposition of a Fock state. The Fock state of the electromagnetic field is very difficult to produce in experiments. Nevertheless, these states are very important in quantum optics because of their intrinsic quantum nature. An interesting situation with coherent-state field will be discussed in the following.

We note that distinct patterns for the atomic information entropy as well as atomic inversion will arise, depending on the values of $\gamma(t)$, η and $\overline{n}$. In figure 3, we show the dynamical evolution of a three-level trapped ion under the coherent superposition of its states. In spite of these successes, a closed analytical description of the collapse-revival pattern has so far proved to be elusive; however, an elegant approximation scheme valid for a number of initial conditions has been presented in [31], improving the earlier work of [32]. Among other things, they have demonstrated that when the ion is initially completely excited or de-excited, and the initial photon number distribution of the field is sufficiently smooth then the shape of each revival is a direct reflection of the shape of the photon number distribution (see figure 3(d)). This direct relationship can be affected by the presence of initial atomic coherence. It has been noted [33] that if the ion is initially prepared in a coherent superposition of its energy eigenstates, then the revivals can be largely suppressed, effectively freezing the value of the atomic state populations. The information entropies for $\hat{\sigma}_x$, $\hat{\sigma}_y$ in this case attain its minimum at the half of revival time. They show the same trend, however in the limit fluctuations in $H(\hat{\sigma}_y)$ are higher. The variable $H(\hat{\sigma}_z)$ does not show minimum, during the collapse time and the oscillations occur during the revival period only as can be seen from

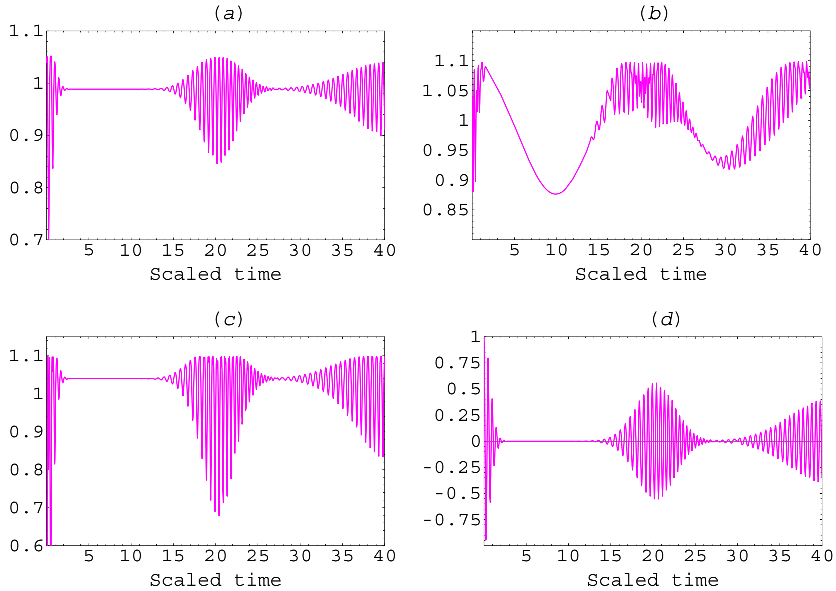

Figure 3. Time dependence of (*a*) $H(\hat{\sigma}_x)$, (*b*) $H(\hat{\sigma}_y)$, (*c*) $H(\hat{\sigma}_z)$ and (*d*) the atomic inversion $\langle\sigma_z\rangle = \rho_{cc} - \rho_{bb} - \rho_{aa}$ for the initial average photon number $\bar{n} = 20, \eta = 0.202$ and time-independent ion–field coupling $\gamma(t) = \gamma$.

figure 3(*a*). Note that the presented model describes the physical situation where all the three levels are equally correlated with each other. However, the model can easily be extended to study a variety of other situations.

In order to appreciate some different situations, we show, in what follows, some plots of the information entropies $H(\hat{\sigma}_\alpha)$ for different values of the ion–field coupling. As might be expected, the behaviour of the present three-level system changes dramatically depending on the initial field state, the ion–field couplings and the Lamb–Dicke parameter. Using the coherent state as an initial state of the field, the dependence of the information entropies on the scaled time γt when we set $\gamma(t) = \gamma \sin(\varpi t)$ is shown in figure 4. It is easy to realize in this case that the period to reach the maximum value of the information entropies is shorter than that for the time-independent case. As time goes on we see more fluctuations showing weak entanglement from time to time. This phenomenon gets more pronounced when we increase the value of the Lamb–Dicke parameter too, and the information entropy values in this case are decreased. The regularity here is pronounced as in the case of figures 1(*b*) and 2(*b*) where the time-dependent case was considered.

It should be emphasized that our results (see figures 2 and 3) are stated and proved in language accessible for quantum information theory. In particular, we can say that the quantum information entropy $H(\hat{\sigma}_\alpha)$ can be used as a new version of the entanglement measures, stating necessary and sufficient conditions for a functional to coincide with the reduced von Neumann entropy on pure states.

It is rather interesting to note that entropic uncertainty relations and entropy squeezing play a crucial role in quantum computation as well. However, works dealing with the entropy squeezing have been limited to the two-level systems [34]. Open questions remain, including how to find suitable forms of the relation between entropic uncertainty relations and the atomic information entropy in the three-level systems [35–39]. The basic idea of this approach is to

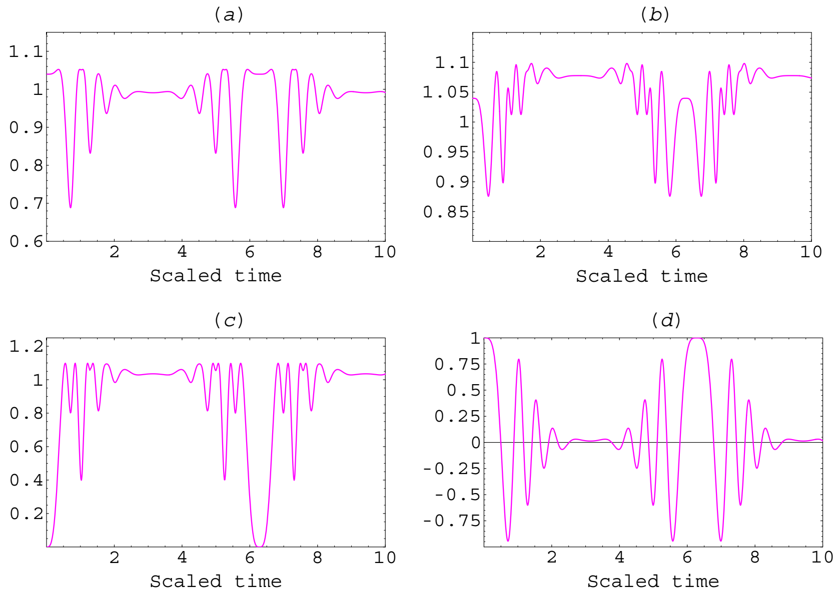

Figure 4. The same as figure 3 but the ion–field couplings are taken to be time dependent $\gamma(t) = \gamma\,\sin(\varpi t)$.

replace the statistical variance with the Shannon entropy as an estimator of the uncertainties associated with the measurement process. An analysis of the squeezing together with many other aspects of the information entropy in multi-level systems, such as entropy squeezing, entropic uncertainty relations, etc will be discussed in another publication.

In our treatment we have focused on the single-laser field. It would thus be interesting to study the non-degenerate two-photon process. We could imagine having a transition in which one photon is visible, and the other is, say, infrared. The frequencies of these two photons could be chosen in such a way that we would obtain a large two-photon coupling and hence this model would be easier to realize. We hope to report on such issues in a forthcoming paper. Also, one could use the well-established field of ion trapping as a testing ground for strongly coupled QED because for a trapped ion the coupling parameters can be varied by the laser field strength. The ability to control/vary the coupling is an attractive feature of the trapped-ion system.

3. Conclusions

The present work has been devoted to a detailed analysis of both analytical and numerical investigations of the process of quantum information entropy in the three-level systems. We have completely solved the problem in the case of three-level system, and found an extension in the case where ion–field coupling is time dependent. Treating the time-dependent ion–field interaction to manipulate the quantum information entropy has many new and important dynamical quantities. In fact, with the recent experimental results we can foresee no fundamental obstacle to build a scalable quantum computer with trapped ions. Of course, technical development may impose severe restrictions on the time scale in which this is achieved. Although the present paper concentrates on the analysis of a three-level system, the present methods can be applied to multi-level systems, for which simple results can be obtained. The quantum information entropy process of the model is very rich and much more can be learned from specific features of the Shannon entropy and quantum field entropy such as its periodic oscillations. We have found an intimate connection between these information entropies and entanglement. Specifically, we have pointed out the very special role played by the time-dependent ion–field coupling on the behaviour of the entropy forms. This is in accordance with the known fact that the entanglement properties are dictated by the entropy of the subsystem.

From a very fundamental point of view, we have discovered a new feature of the quantum information entropy, which shows how far the quantum field entropy lies from information entropy in the pure state case. A particularly interesting aspect of our work is the introduction of the ansatz which results from the observation that one can use the eigenvalues and eigenvectors of the atomic operators to define the quantum information entropy in the multi-level systems.

The general forms of the information entropy of the three-level trapped ions taking into account a time-dependent ion–field interaction are clearly exhibited and they are new as far as we are aware.

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